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Convergence of the (GOP) Algorithm for a Large Class of Smooth Optimization Problems

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Abstract. In this paper it is shown that the (GOP) algorithm is guaranteed to be convergent for a large class of smooth mathematical programming problems.

Keyword: GOP algorithm.

1. Introduction

Global optimization of nonconvex programming problems has been an important topic in optimization theory and has generated significant interest in recent years. A new primal-relaxed dual method, called the GOP algorithm, is reported to be efficient for bilinear programming problems, polynomial programming problems and rational polynomial programming problems (see, for example,[1]-[2], and [5]-[7]). This method can be applied when the problem has been formulated in the following standard form:

$$\min_{x,y} f(x,y) \tag{SOP}$$

subject to $g_i(x,y) \leq 0, h_j(x,y) = 0, x \in X, y \in Y$
with $1 \leq i \leq k, 1 \leq j \leq p$,

where X and Y are non-empty compact convex sets in R^n and R^m $(n, m \geq 1)$, $f(\cdot, y), g_i(\cdot, y), f(x, \cdot)$ and $g_i(x, \cdot)$ are differentiable convex functions for any fixed $y \in Y$ or $x \in X$, and $h_j(x, \cdot)$ and $h_j(\cdot, y)$ are affine for any fixed x in X or y in Y. The GOP method was not considered applicable to very broad mathematical programming problems until the result in [3]. In [3] we proved that a large class of smooth mathematical programming problems can indeed be reformulated in this form by a simple transformation of variables. We now briefly state the result in [3]. Let X be a non-empty compact convex set in R^n. Let $F(\cdot)$ and $G_i(\cdot)$ $(1 \leq i \leq k)$ be continuous function on X. We will assume that $F, G_i \in C^2(R^N)$. We now consider the following optimization problem:

$$\min_x F(x) \qquad (GP)$$

subject to $G_i(x) \leq 0, \quad x \in X, \quad \text{with} \quad 1 \leq i \leq k.$

It is clear that (GP) represents a large class of mathematical programming problems. In [3] we proved the following theorem:

THEOREM 1. *Let X, F and $G_i(1 \leq i \leq k)$ satisfy the conditions in (GP). Then the (GP) can be equivalently formulated in the following standard form:*

$$\min_{x,y} f(x, y) \qquad (ROP)$$

subject to $g_i(x, y) \leq 0, h_j(x, y) = 0, x \in X, y \in Y$

with $1 \leq i \leq k, 1 \leq j \leq k,$

where f, g_i and h_j satisfy the all conditions in (SOP).

The functions f, g_i and h_j can be explicitly given as $f(x, y) = F(x) + \alpha x^T x - \alpha y^T x$, $g_i(x, y) = G_i(x) + \alpha x^T x - \alpha y^T x$ for $1 \leq i \leq k$ and $h_j(x, y) = x_j - y_j$ for $1 \leq j \leq k$. The constant α can be given by estimating the eigenvalues of the Hessian matrices of F and G_i $(1 \leq i \leq k)$. From this result it is clear that the GOP method is actually applicable to very broad mathematical programming problems. Most of useful finite dimensional problems in practice are virtually covered. We refer the readers to [5] and [8]-[9] for important applications of this difference of convex functions' transformation.

It is not straightforward to apply the convergence theorem in [1] and [2] to (ROP). The original conditions given in [1] and [2] are that the Slater's constraint qualification holds for (ROP) for any fixed y and the optimal multipliers of (ROP) for any fixed y are uniformally bounded. It is clear that these conditions are too restrictive to apply here. For example, in (ROP) if $x_0 \in X$ is such that $G(x_0) = 0$ for an i, then the Slater's constraint qualification will not hold at $y = x_0$. The verification of the boundedness condition on the optimal multipliers is also very difficult for a complicated problem. In this paper, we will show that the (GOP) algorithm is guaranteed to be convergent when applied to (ROP).

2. A Generalized Primal-Relaxed Dual Approach

In this section we will describe the (GOP) algorithm in a general framework in which one can examine some important issues concerning the (GOP), as well as other alternative primal-relaxed dual algorithms. For instance, we can easily show the convergence of the (GOP) when applied to (ROP). This idea has also been used in [4] to develop and examine new primal and relaxed dual algorithms. We begin with a lemma:

LEMMA 1. *Let f, g and h be given as in (ROP). Then for any $y_0 \in V$ there are $(\lambda_0, \mu_0) \in R^p \times R^k_+$ and $x_0 \in X$ such that $\mu_0^T g(x_0, y_0) = 0, (x_0, y_0) \in AD$, and for any $x \in X$ and $(\lambda, \mu) \in R^p \times R^k_+$*

$$L(x_0, y_0, \lambda, \mu) \le L(x_0, y_0, \lambda_0, \mu_0) \le L(x, y_0, \lambda_0, \mu_0), \qquad (SP)$$

where

$$L(x, y, \lambda, \mu) = f(x, y) + \lambda^T h(x, y) + \mu^T g(x, y). \qquad (2)$$

AD is the admissible set of (ROP) which is assumed to be compact, and $V = \{y \in Y$: there is an $x \in X$ such that $g(x, y) \le 0$ and $h(x, y) = 0.\}$.

Proof. Based on the convexity of $f(\cdot, y_0)$, it is easy to see that $x_0 = y_0, \lambda_0 = -\nabla_x f(x_0, y_0)$ and $\mu_0 = 0$ will satisfy the requirement of the lemma.

We now give the primal-relaxed dual formula for (ROP). First it is clear that for any $y_0 \in V = \{y \in Y$: there is an $x \in X$ such that $g(x, y) \le 0$ and $h(x, y) = 0.\}$,

$$\nu = \min_{(x,y)\in AD} f(x, y) \le \min_{x \in X, g(x,y_0) \le 0, h(x,y_0)=0} f(x, y_0) = \nu^+(y_0). \qquad (P)$$

On the other hand, it follows from the convexity of $f(\cdot, y)$ and $g_i(\cdot, y)$ and the left inequalities in (SP) that

$$\begin{aligned} \nu = \min_{(x,y)\in AD} f(x, y) &= \min_{y\in V} \min_{x\in X, g(x,y)\le 0, h(x,y)=0} f(x, y) = \\ &\min_{y\in V} \max_{\lambda,\mu\ge 0} \min_{x\in X} (f(x, y) + \lambda^T h(x, y) + \mu^T g(x, y)) \ge \\ \min_{y\in V} \max_{(\lambda_t,\mu_t)\in U} \min_{x\in X} H^{(\lambda_t,\mu_t)}(x, y) &= \nu^-(U, H), \end{aligned} \qquad (RD)$$

where U is a sequence $\{(\lambda_t, \mu_t)\}$ with $(\lambda_t, \mu_t) \in R^p \times R^k_+$ for t=1,2,...,N and the mapping $H : U \to C^1(X \times Y)$ is such that that $H^{(\lambda_t,\mu_t)}$ is a continuous function on $X \times Y$ satisfying that $f + \lambda^T h + \mu^T g \ge H^{(\lambda_t,\mu_t)}$ on $X \times Y$ for every fixed

$(\lambda_t, \mu_t) \in U$.

The primal-relaxed dual approach is to find a sequence of $y_n \in Y$ and U_n, and a rule H_n to give $H_n^{(\lambda_t,\mu_t)}$ for every $(\lambda_t, \mu_t) \in U_n$ such that $\nu^+(y_n) - \nu^-(U_n, H_n) \to 0$ as $n \to \infty$. The selections of U_n and H_n are clearly not unique but they must make it possible to find the global solutions computationally for (RD). Let (x_i, λ_i, μ_i) be a solution of (SP) for a given y_j. We will take

$$H^{(\lambda_i,\mu_i)}(x, y) = L(x_i, y_i, \lambda_i, \mu_i) + \nabla_x L(x_i, y, \lambda_i, \mu_i)(x - x_i) + \nabla_y L(x_i, y_i, \lambda_i, \mu_i)(y - y_i). \quad (3)$$

The resulting algorithm is the (GOP) (see [1] and [2]), and has the following generic steps:
(1) Given $y_0 \in V$ and $\varepsilon > 0$.
(2) Given y_n and solve (SP) for $y = y_n$ to obtain x_n and (λ_n, μ_n).
(3) Solve (RD) to obtain y_{n+1}, where $U = \{(\lambda_i, \mu_i)\}$ $i = 1, ..., n$ and $H_n^{(\lambda_i,\mu_i)}$ is given by equation (2).
(4) If $\nu^+(y_n) - \nu^-(U_n, H_n) \leq \varepsilon$, stop. Otherwise go to step 2.
There are other useful forms of H_n which lead to different primal-relaxed dual algorithms (see [4]).

3. Convergence of (GOP) for (ROP)

Now we can show the convergence of the (GOP) when applied to (ROP).

THEOREM 2. *Assume that for any $y \in V$ there exists a solution SP(y) of SP such that the mapping of y to $SP(y)$ is locally bounded on V. Let $\{(x_i, y_i)\}$ be generated from the (GOP) algorithm applied to (ROP). Then every cluster point of $\{(x_i, y_i)\}$ is a solution of (ROP). Moreover, the (GOP) terminates at finite steps.*

Proof. It follows from the compactness of the admissible set of (ROP) that there is $\{k_i\}$ a subsequence of $\{i\}$ such that $(x_{k_i}, y_{k_i}) \to (x_0, y_0)$ which is admissible. We show that (x_0, y_0) is a global solution of (ROP). By noting that $(x_{k_i}, \lambda_{k_i}, \mu_{k_i})$ is a solution of (SP) for the fixed y_{k_i} one can see that $\nu^+(y_{k_i}) = L(x_{k_i}, y_{k_i}, \lambda_{k_i}, \mu_{k_i})$. On the other hand, one can show that

$$\nu^-(U_{k_{i+1}-1}, H_{k_{i+1}-1}) \geq \quad L(x_{k_i}, y_{k_i}, \lambda_{k_i}, \mu_{k_i}) - \delta_{k_i}, \quad (4)$$

with $\delta_{k_i} \geq 0$ and $\delta_{k_i} \to 0$ as $i \to \infty$, because $\nabla_x L(x_{k_i}, y_{k_i}, \lambda_{k_i}, \mu_{k_i})(x - x_{k_i}) \geq 0$ for $x \in X$, X is compact space, $\{(\lambda_i, \mu_i)\}$ is bounded by the assumption stated in the theorem, and *f* and *g* are C^2 functions. Note that $\{\nu^-(U_i, H_i)\}$ is bounded above and is increasing as $i \to \infty$ so that $\{\nu^-(U_i, H_i)\}$ is convergent as $i \to \infty$. Thus $0 \leq \nu^+(y_{k_i}) - \nu^-(U_{k_i}, H_{k_i}) \leq \delta_{k_i} + \nu^-(U_{k_{i+1}-1}, H_{k_{i+1}-1}) - \nu^-(U_{k_i}, H_{k_i}) \to 0$

as $i \to \infty$. Therefore $\nu^+(y_{k_i}) \to \min_{(x,y)\in AD} f(x,y)$, where AD is the admissible set of (ROP), as $\nu^+(y_i)$ is a upper bound of (ROP) and $\nu^-(U_i, H_i)$ is a lower one for any $\{i\}$. By noting that $\nu^+(y_i) = f(x_i, y_i)$ and the continuity of f one can see that (x_0, y_0) is a global solution of (ROP) since $(x_{k_i}, y_{k_i}) \to (x_0, y_0)$.
It follows from the proof above that for any subsequence of $\{i\}$ there is a subsequence of this subsequence, still denoted as $\{i\}$, such that $\nu^+(y_i) - \nu^-(U_i, H_i) \to 0$ and $\nu^+(y_i) \to \min_{(x,y)\in AD} f(x,y)$, which is unique. Consequently $\nu^+(y_i) - \nu^-(U_i, H_i) \to 0$ and $\nu^+(y_i) \to \nu$ as $i \to \infty$.

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